Eur. Phys. J. B **47**, 291–293 (2005)
DOI: 10.1140/epjb/e2005-00330-1

THE EUROPEAN PHYSICAL JOURNAL B

On an integral equation arising in the transport of radiation through a slab involving internal reflection

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Received 23 May 2005 / Received in final form 22 July 2005
Published online 11 October 2005 –

Abstract. The integral equation derived by Nieuwenhuizen and Luck for transmission of radiation through an optically thick diffusive medium is reconsidered in the light of radiative transfer theory and extended to slabs of arbitrary thickness.

PACS. 05.60.Cd Classical transport

1 Introduction

The transmission of radiation through turbid media has been of considerable interest for many years. It is assuming even more importance because of the development of tissue scanning devices which employ infra-red radiation. The interpretation of the scans requires ever more sophisticated models and mathematical procedures. One problem which has exercised workers in this field is the validity of diffusion theory. Whilst it can be shown that diffusion theory is reasonably accurate in the bulk of media which are many mean free paths thick, problems arise near boundaries and interfaces where radiation fluxes can vary rapidly over the distance of a few mean free paths. Special boundary conditions have been devised to correct diffusion theory but even these methods fail in certain situations. In such cases a full transport treatment is required. These matters are not confined to radiative transfer and have been met for many years in neutron transport theory and the associated problem of nuclear reactor design [1,2]. However, radiation problems suffer from a further source of uncertainty, namely because of the transition from the electromagnetic equations of Maxwell [3] to those of radiative transfer [4]. A definitive paper showing how Maxwell's equations are related to radiative transfer theory was given by Fante [5] and a more practical approach is described by Nieuwenhuizen and Luck [6], see also [7,8]. In their work Nieuwenhuizen and Luck have shown how, for optically thick slabs, i.e. where the mean free path of the photons is very much less than the thickness of the slab, the electromagnetic solution can be linked to a classical radiative transfer equation. This is a significant advance and it enables one to see the wave-photon transition very clearly. The essence of the solution is to construct the mean amplitude of the Green's function $G(\mathbf{r},\mathbf{r}')$ in the medium using the wave equation. Then the ladder approximation for the Bethe-Salpeter equation [9] for light propagating through a multipli-scattering medium is used to derive an equation for the radiation flux $\Phi(\mathbf{r})$ in the form

$$\Phi(\mathbf{r}) = \frac{4\pi}{\ell}\left[I_{in}(\mathbf{r}) + \int_V d\mathbf{r}' \left|G(\mathbf{r},\mathbf{r}')\right|^2 \Phi(\mathbf{r}')\right] \tag{1}$$

where ℓ is a mean free path and $I_{in}(\mathbf{r})$ is the source term from incident light on the surface of the body of volume V. Equation (1) may be shown to be equivalent to the classic form for radiative transfer with isotropic scattering.

Nieuwenhuizen and Luck then consider equation (1) for a slab of thickness a and define the optical path $\tau = z/\ell$ and obtain the Green's function $G(z,z')$ for an optically thick slab. They further set $\Phi(\mathbf{r}) = C_0\Gamma(\tau)$, where C_0 is a scaling factor, and reduce equation (1) to the form

$$\Gamma(\tau) = e^{-\tau/\mu_0} + \int_0^a d\tau' M(\tau,\tau')\,\Gamma(\tau') \tag{2}$$

where $\vartheta_0 \quad (\mu_0 = \cos\vartheta_0)$ is the angle with respect to the normal at which the incident beam strikes the plane $\tau = 0$. The optical thickness of the slab is a and

$$M(\tau,\tau') = M_B(\tau,\tau') + M_L(\tau,\tau') + M_L(a-\tau, a-\tau') \tag{3}$$

with

$$M_B(\tau,\tau') = \frac{1}{2}\int_0^1 \frac{d\mu}{\mu}\, e^{-|\tau-\tau'|/\mu} = \frac{1}{2}E_1(|\tau-\tau'|) \tag{4}$$

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$$M_L(\tau,\tau') = \frac{1}{2}\int_0^1 \frac{d\mu}{\mu} R(\mu)\, e^{-(\tau+\tau')/\mu} \tag{5}$$

where $R(\mu)$ is the internal reflection coefficient on the surfaces $\tau = 0$, $\tau = a$. A further restriction is that the medium is non-absorbing.

We wish to extend the above formalism to cover the case of slabs of arbitrary thickness. This can be done by considering the problem from the point of view of radiative transfer with appropriate boundary conditions.

2 Theory

We may write the equation of radiative transfer for the angular photon flux $\phi(\tau,\mu)$ for isotropic scattering as [4],

$$\mu\frac{\partial\phi(\tau,\mu)}{\partial\tau} + \phi(\tau,\mu) = \frac{c}{2}\int_{-1}^{1} d\mu'\phi(\tau,\mu') \equiv \frac{c}{2}\phi_0(\tau) \tag{6}$$

where $c = \Sigma_s/(\Sigma_s+\Sigma_a)$. The scattering cross section is defined by Σ_s and the absorption cross section by Σ_a.

The boundary conditions associated with equation (6) which describe internal reflection are

$$\phi(0,\mu) = R(\mu)\,\phi(0,-\mu) + q(\mu); \qquad \mu > 0 \tag{7a}$$

$$\phi(a,\mu) = R(-\mu)\,\phi(a,-\mu); \qquad \mu < 0 \tag{7b}$$

with $R(\mu)$ the reflection coefficient as in equation (5) and $q(\mu)$ the incident distribution at $\tau = 0$. Equation (6) may be integrated to yield

$$\phi(\tau,\mu) = \phi(0,\mu)\,e^{-\tau/\mu} + \frac{c}{2\mu}\int_0^{\tau} d\tau'\phi_0(\tau')\,e^{-(\tau-\tau')/\mu}; \quad \mu > 0 \tag{8a}$$

$$\phi(\tau,\mu) = \phi(a,\mu)\,e^{(a-\tau)/\mu} - \frac{c}{2\mu}\int_{\tau}^{a} d\tau'\phi_0(\tau')\,e^{(\tau'-\tau)/\mu}; \quad \mu < 0. \tag{8b}$$

Thus we find for $\mu > 0$

$$\phi(a,\mu) = \phi(0,\mu)\,e^{-a/\mu} + \frac{c}{2\mu}\int_0^{a} d\tau'\phi_0(\tau')\,e^{-(a-\tau')/\mu} \tag{9a}$$

$$\phi(0,-\mu) = \phi(a,-\mu)\,e^{-a/\mu} + \frac{c}{2\mu}\int_0^{a} d\tau'\phi_0(\tau')\,e^{-\tau'/\mu}. \tag{9b}$$

Using the boundary conditions (7a, b), we have

$$\phi(a,\mu) = (R(\mu)\phi(0,-\mu) + q(\mu))\,e^{-a/\mu} + \frac{c}{2\mu}\int_0^{a} d\tau'\phi_0(\tau')\,e^{-(a-\tau')/\mu} \tag{10a}$$

$$\phi(0,-\mu) = R(\mu)\,\phi(a,\mu)\,e^{-a/\mu} + \frac{c}{2\mu}\int_0^{a} d\tau'\phi_0(\tau')\,e^{-\tau'/\mu}. \tag{10b}$$

Solving (10a, b) for $\phi(a,\mu)$ and $\phi(0,-\mu)$ gives

$$\phi(a,\mu) = \frac{1}{1-R^2(\mu)e^{-2a/\mu}}\cdot\Bigg[q(\mu)e^{-a/\mu} + \frac{c}{2\mu}\int_0^{a} d\tau'\phi_0(\tau')\Big(R(\mu)e^{-(a+\tau')/\mu} + e^{-(a-\tau')/\mu}\Big)\Bigg] \tag{11a}$$

$$\phi(0,-\mu) = \frac{1}{1-R^2(\mu)e^{-2a/\mu}}\cdot\Bigg[q(\mu)R(\mu)e^{-2a/\mu} + \frac{c}{2\mu}\int_0^{a} d\tau'\phi_0(\tau')\Big(e^{-\tau'/\mu} + R(\mu)e^{-(2a-\tau')/\mu}\Big)\Bigg]. \tag{11b}$$

Inserting these expressions into (8a, b) and integrating over $\mu\,(-1, 1)$, we find the following integral equation for $\phi_0(\tau)$,

$$\phi_0(\tau) = \int_0^1 d\mu q(\mu)\frac{e^{-\tau/\mu} + R(\mu)e^{-(2a-\tau)/\mu}}{1-R^2(\mu)\,e^{-2a/\mu}} + \frac{c}{2}\int_0^{a} d\tau'\phi_0(\tau')\,(E_1(|\tau-\tau'|) + K(\tau,\tau')) \tag{12}$$

where

$$K(\tau,\tau') = \int_0^1 \frac{d\mu R(\mu)}{\mu(1-R^2(\mu)e^{-2a/\mu})}\Bigg\{e^{-(\tau+\tau')/\mu} + e^{-(2a-\tau-\tau')/\mu} + R(\mu)\Big(e^{-(2a+\tau-\tau')/\mu} + e^{-(2a-\tau+\tau')/\mu}\Big)\Bigg\}. \tag{13}$$

It is this equation with $\phi_0(\tau) = \Gamma(\tau)$, $q(\mu) = \delta(\mu-\mu_0)$, $c = 1$, that should agree with equation (2). Obviously it does not. However, we notice that if we neglect the terms of $O\left(e^{-2a}\right)$ in equation (12) we arrive at equation (2). This is the optically thick limit and enables us to make contact with the equation of Nieuwenhuizen and Luck [6]. Later in their work, Nieuwenhuizen and Luck extend the equation to include the transverse wave number $\mathbf{q}$. Such modifications are readily incorporated into equations (6)–(13) and indeed the appearance of the kernels is analogous to those developed many years ago to deal with the neutron leakage from a gap in a nuclear reactor [1,10].

Two special cases of the above theory are worth noting. If we set $R = 0$ in equation (12), then it reduces to the classical case [1]

$$\phi_0(\tau) = \int_0^1 d\mu q(\mu)\, e^{-\tau/\mu} + \frac{c}{2}\int_0^a d\tau' \phi_0(\tau') E_1(|\tau - \tau'|) \qquad (14)$$

and, for $a \to \infty$, i.e. a semi-infinite half-space, we have the albedo problem, viz.:

$$\phi_0(\tau) = \int_0^1 d\mu q(\mu)\, e^{-\tau/\mu} + \frac{c}{2}\int_0^\infty d\tau' \phi_0(\tau') \times \left[E_1(|\tau - \tau'|) + \int_0^1 \frac{d\mu}{\mu} R(\mu)\, e^{-(\tau+\tau')/\mu}\right]. \qquad (15)$$

This equation has recently been solved using the Wiener-Hopf technique [11].

In many practical cases, it is the emergent surface angular distributions $\phi(0, -\mu)$ and $\phi(a, \mu)$ which are of interest. These can be obtained from a solution of equation (12) and insertion into (11a, b). There is also a method for obtaining $\phi(0, -\mu)$ and $\phi(a, \mu)$ as a pair of coupled singular integral equations. Whilst these are more elegant mathematically, from a practical point of view it is easier to use (11a, b) for numerical calculations.

3 Conclusions

Use of the correct reflective boundary conditions in the framework of radiative transfer theory extends the results of earlier workers [6–8] to transmission through slabs of arbitrary thickness. It would be of considerable interest to see whether a solution of the wave equation could be formulated which took into account the interaction between the two boundaries at $\tau = 0$ and a since this would most likely account for the missing terms.

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